RANGE-BASED EGARCH OPTION PRICING MODELS[1]

Introduction

The research in this paper is focused on the innovative range-based volatility models introduced in Alizadeh, Brandt, and Diebold (2002) (hereafter ABD). We develop new option pricing models using multi-factor diffusion approximations couched within this theoretical framework and examine their properties in comparison with the traditional Black-Scholes model.

We assume that the log stock price s follows a drift-less Brownian motion $\delta\sigma = \sigma\delta\Omega$. The volatility of daily log returns, denoted $h = \sigma/\sqrt{252}$, is assumed constant within each day, at ht from the beginning to the end of day t, but is allowed to change from one day to the next, from ht at the end of day t to ht+1 at the beginning of day t+1. Under these assumptions, ABD show that the log range, defined as:

$$D_t = \ln\left(\max_{\tau\in[t,t+1]} s_\tau - \min_{\tau\in[t,t+1]} s_\tau\right) \tag{4.1}$$

is to a very good approximation distributed as

$$D_t \sim N\left[0.43 + \ln h_t, 0.29^2\right] \tag{4.2}$$

where N[*m; v*] denotes a Gaussian distribution with mean *m* and variance *v*. The above equation demonstrates that the log range is a *noisy* linear proxy of *log* volatility ln *ht*. By contrast, according to the results of Alizadeh, Brandt,and

[1] The author wishes to acknowledge the contribution to the research in this chapter due to the many helpful comments and suggestions from Professor Stephen Taylor of the University of Lancaster.

Diebold (2002), the log absolute return has a mean of 0.64 + ln ht and a variance of 1.11. However, the distribution of the log absolute return is far from Gaussian. The fact that both the log range and the log absolute return are linear log volatility proxies (with the same loading of one), but that the standard deviation of the log range is about one-quarter of the standard deviation of the log absolute return, makes clear that the range is a much more informative volatility proxy. It also makes sense of the finding of Andersen and Bollerslev (1998) that the daily range has approximately the same informational content as sampling intra-daily returns every four hours.

It is well known that the range suffers from a discretization bias because the highest (lowest) stock price observed at discrete points in time is likely to be lower (higher) than the true maximum (minimum) of the underlying diffusion process. It follows that the observed range is a downward-biased estimate of the true range (which in turn is a noisy proxy of volatility). Rogers and Satchell (1991) devise a correction of the observed range that virtually eliminates this bias. However in many cases the discretization bias is not likely to be a problem where the underlying asset is very liquid and the time that elapses between trades (recorded prices) is negligible.

Except for the model of Chou (2001), GARCH-type volatility models rely on squared or absolute returns (which have the same information content) to capture variation in the conditional volatility h_t. Since the range is a more informative volatility proxy, it makes sense to consider range-based GARCH models, in which the range is used in place of squared or absolute returns to capture variation in the conditional volatility. This is particularly true for the EGARCH framework of Nelson (1990), which describes the dynamics of log volatility (of which the log range is a linear proxy).

ABD consider variants of the EGARCH framework introduced by Nelson (1990). In general, an EGARCH(1,1) model performs comparably to the GARCH(1,1) model of Bollerslev (1987). However, for stock indices the in-sample evidence reported by Hentschel (1995) and the forecasting performance presented by Pagan and Schwert (1990) show a slight superiority of the EGARCH specification. One reason for this superiority is that EGARCH models can accommodate asymmetric volatility (often called the "leverage effect," which refers to one of the explanations of asymmetric volatility), where increases in volatility are associated more often with large negative returns than with equally large positive returns.

The one-factor range-based model (REGARCH 1) takes the form:

$$\ln h_t - \ln h_{t-1} = k_h(\vartheta - \ln h_{t-1}) + \phi_h X_{t-1}^D + \delta_h R_{t-1}/h_{t-1} \tag{4.3}$$

where the returns process R_t is conditionally Gaussian: $R_t \sim N[0, h_t^2]$

and the process innovation is defined as the standardized deviation of the log range from its expected value:

$$X_{t-1}^D = (D_{t-1} - 0.43 - \ln h_{t-1})/0.29 \tag{4.4}$$

Following Engle and Lee (1999), ABD also consider multi-factor volatility models. In particular, for a two-factor range-based EGARCH model (REGARCH2), the conditional volatility dynamics) are as follows:

$$\ln h_t - \ln h_{t-1} = k_h(\ln q_{t-1} - \ln h_{t-1}) + \phi_h X_{t-1}^D + \delta_h R_{t-1}/h_{t-1} \tag{4.5}$$

$$\ln q_t - \ln q_{t-1} = k_q(\vartheta - \ln q_{t-1}) + \phi_q X_{t-1}^D + \delta_q R_{t-1}/h_{t-1} \tag{4.6}$$

where ln q_t can be interpreted as a slowly-moving stochastic mean around which log volatility ln h_t makes large but transient deviations (with a process determined by the parameters κ_h, ϕ_h and δ_h).

The parameters θ, κ_q, ϕ_q and δ_q determine the long-run mean, sensitivity of the long run mean to lagged absolute returns, and the asymmetry of absolute return sensitivity respectively.

The intuition is that when the lagged absolute return is large (small) relative to the lagged level of volatility, volatility is likely to have experienced a positive (negative) innovation. Unfortunately, as we explained above, the absolute return is a rather noisy proxy of volatility, suggesting that a substantial part of the volatility variation in GARCH-type models is driven by *proxy noise* as opposed to true information about volatility. In other words, the noise in the volatility proxy introduces noise in the implied volatility process. In a volatility forecasting context, this noise in the implied volatility process deteriorates the quality of the forecasts through less precise parameter estimates and, more importantly, through less precise estimates of the current level of volatility to which the forecasts are anchored.

The REGARCH Framework

Discrete-time ARCH models can be approximated by diffusions (and vice versa) by following methods in "ARCH models as diffusion approximations" by Daniel Nelson, 1990, J Econometrics 45, 7-38. The basic idea is that the means and variances of the processes can be matched.[2]

[2] *Nelson's 1990 paper is demanding mathematics. Some of the same ideas appear in the authoritative survey paper : "ARCH models" by T. Bollerslev, R.F. Engle and D.B. Nelson, 1994, in : Handbook of Econometrics, volume IV (North-Holland, Amsterdam), 2959-3038.*

The 2-factor range model defines conditional standard deviations (not variances) for one-day returns and hence these quantities are not annualized.

Let t measure time in years. Also suppose data is collected once every s years, so for daily data $s = 1/252$ approximately.

The discrete-time model is then:

$$R_t = h_t Z_t^{(1)}, \quad Z_t^{(1)} \sim N(0,1),$$

$$\ln(h_t) - \ln(h_{t-s}) = \kappa_h(\ln(q_{t-s}) - \ln(h_{t-s})) + \varepsilon_{t-s}, \tag{4.7}$$

$$\ln(q_t) - \ln(q_{t-s}) = \kappa_q(\theta - \ln(q_{t-s})) + \eta_{t-s}, \tag{4.8}$$

with residuals

$$\varepsilon_t = \phi_h X_t^D + \delta_h R_t / h_t, \tag{4.9}$$

$$\eta_t = \phi_q X_t^D + \delta_q R_t / h_t . \tag{4.10}$$

The latter are i.i.d. residuals, that are shocks to ln(h) and ln(q). The variances of these residuals depend on the parameters that define them. In the following equations, the residuals are scaled to make them products of their standard deviations and standardized residuals. Also, the volatility terms are annualized – this is helpful for comparisons with familiar diffusions, but it is also necessary to obtain terms on the same scale so that limits can be considered.

Now substitute

$$h_t = \sqrt{s} h_t^* \text{ and } q_t = \sqrt{s} q_t^*, \tag{4.11}$$

so the starred variables are annualized standard deviations. Also let

$$\varepsilon_t = \gamma_h \sqrt{s} Z_t^{(2)} \text{ and } \eta_t = \gamma_q \sqrt{s} Z_t^{(3)} \tag{4.12}$$

with all variables Z standardized, i.e. mean zero and variance one. Then

$$\ln(h_t^*) - \ln(h_{t-s}^*) = \frac{\kappa_h}{s}(\ln(q_{t-s}^*) - \ln(h_{t-s}^*))s + \gamma_h \sqrt{s} Z_{t-s}^{(2)}, \tag{4.13}$$

$$\ln(q_t^*) - \ln(q_{t-s}^*) = \frac{\kappa_q}{s}(\theta - \tfrac{1}{2}\ln(s) - \ln(q_{t-s}^*))s + \gamma_q \sqrt{s} Z_{t-s}^{(3)}. \tag{4.14}$$

We can also add the equation

$$\ln(P_t) - \ln(P_{t-s}) = h_t^* \sqrt{s} Z_t^{(1)}. \tag{4.15}$$

The three equations then define a model M(s) for shocks defined once every s years.

Nelson's paper focuses on the limit of a family of models M(u) as u decreases towards zero. To obtain M(u) we retain the "drift rates" in M(s), thus :

$$\ln(P_t) - \ln(P_{t-u}) = R_t = h_t^* \sqrt{u} Z_t^{(1)}, \tag{4.16}$$

$$\ln(h_t^*) - \ln(h_{t-u}^*) = \frac{\kappa_h}{s}(\ln(q_{t-u}^*) - \ln(h_{t-u}^*))u + \gamma_h \sqrt{u} Z_{t-u}^{(2)}, \tag{4.17}$$

$$\ln(q_t^*) - \ln(q_{t-u}^*) = \frac{\kappa_q}{s}(\theta - \tfrac{1}{2}\ln(s) - \ln(q_{t-u}^*))u + \gamma_q \sqrt{u} Z_{t-u}^{(3)}. \tag{4.18}$$

There are similarities here with equation (3.23) of Nelson (1990).

Every $\sqrt{u}Z$ term corresponds to a term dW in the diffusions, since the variances of $\sqrt{u}Z$ and $W(t) - W(t-u)$ both equal u. The first Z has time subscript t, the second and third Z have subscripts t-u. The distinction does not matter as u approaches zero.

In the diffusion approximation that follows it might be more correct to replace dP/P by $d(\ln P)$. However, we have started by assuming zero mean return which is a simplification anyway and furthermore any option pricing will require the introduction of the risk-free rate into the first equation below if the equations are used in the context of a risk-neutral measure.

The diffusion approximation then involves prices, two annualized volatility components and three Wiener processes:

$$dP = (rP + D)dt + \sigma^{(1)} P \, dW^{(1)}, \tag{4.19}$$

$$d(\ln(\sigma^{(1)}) = \frac{\kappa_h}{s}(\ln(\sigma^{(2)}) - \ln(\sigma^{(1)}))dt + \gamma_h dW^{(2)}, \tag{4.20}$$

$$d(\ln(\sigma^{(2)}) = \frac{\kappa_q}{s}(\theta - \tfrac{1}{2}\ln(s) - \ln(\sigma^{(2)}))dt + \gamma_q dW^{(3)}. \tag{4.23}$$

Here $\sigma^{(1)}$ and $\sigma^{(2)}$ have replaced h^* and q^*, and u has been replaced by dt. We next have to specify the correlations between the Wiener processes. As we

match the covariances of the shocks in the discrete and continuous processes, it is enough to match the correlations of the 3 shocks Z with those of the 3 differentials dW.

There is zero correlation between $Z_t^{(1)}$ and the range variable X_t^D, when these variables are the result of observing a continuous process. Intuitively the correlation is zero because replacing dW by $-dW$ switches the sign of Z but leaves the range the same. This means $Z_t^{(1)}\ X_t^D$ has the same expectation as $-Z_t^{(1)}\ X_t^D$, hence the expectation and correlation are both zero.

Further simplifications occur on the next section because each Z has unit variance.

The correlation matrix of the Wiener differentials is the same as that of the terms. $Z_t^{(i)}$ since the range is uncorrelated with the standardized return,

$$\text{cor}(dW^{(1)}, dW^{(2)}) = \text{cor}(Z_t^{(1)}, Z_t^{(2)}) = \text{cov}(Z_t^{(1)}, \frac{\phi_h X_t^D + \delta_h Z_t^{(1)}}{\gamma_h \sqrt{s}}) = \frac{\delta_h}{\gamma_h \sqrt{s}}. \tag{4.24}$$

Likewise,

$$\text{cor}(dW^{(1)}, dW^{(3)}) = \frac{\delta_q}{\gamma_q \sqrt{s}}. \tag{4.25}$$

$$\mathrm{cor}(dW^{(2)}, dW^{(3)}) = \mathrm{cov}(\frac{\phi_h X_t^D + \delta_h Z_t^{(1)}}{\gamma_h \sqrt{s}}, \frac{\phi_q X_t^D + \delta_q Z_t^{(1)}}{\gamma_q \sqrt{s}}) = \frac{\phi_h \phi_q + \delta_h \delta_q}{\gamma_h \gamma_q s}. \quad (4.26)$$

These expressions are used to derive the correlation matrix between three Wiener processes. This matrix is then decomposed using the Cholesky factorization method described earlier to enable correlated Normal variates to be generated.

The scale terms are given by

$$\gamma_h^2 = \frac{1}{s}(\phi_h^2 + \delta_h^2) \text{ and } \gamma_q^2 = \frac{1}{s}(\phi_q^2 + \delta_q^2) \quad (4.27)$$

providing we are willing to assume that the terms X_{t-1}^D and R_{t-1}/h_{t-1} have unit variance, as supposed by the theory. The assumption, particularly for X_{t-1}^D, can be checked from the time series values given by an estimated REGARCH model.

It transpires that the correlation matrix is singular:

$$|\mathbf{M}| = 1 - \frac{\delta_h^2}{\delta_h^2 + \phi_h^2} - \frac{\delta_q^2}{\delta_q^2 + \phi_q^2} + \frac{\delta_h^2 \delta_q^2}{(\delta_h^2 + \phi_h^2)(\delta_q^2 + \phi_q^2)} - \frac{\phi_h^2 \phi_q^2}{(\delta_h^2 + \phi_h^2)(\delta_q^2 + \phi_q^2)} = 0 \quad (4.28)$$

This means, in effect, that there are only two random factors in the model. However, there are still three state variables required to specify the state of the world (one price and two volatilities).

The explanation is as follows:

In GARCH and Nelson's exponential ARCH, there is one random shock per unit time, which determines the surprise in both the asset price and its volatility. In the

diffusion limit, for exponential ARCH, there are two terms dW() and they are not singular.

For REGARCH with one factor, there are two shocks per unit time (essentially the return and the range) that determine the surprises in the asset price and its volatility.

But, for REGARCH with two factors there are still only two shocks per unit time. This means the three terms Z() are determined by the return and the range. Hence any two Z() give the third one and, indeed, the relationships among the Z() are linear (which makes them singular).

The singularity is therefore a consequence of REGARCH (with two factors) using linear combinations of two shocks to provide the surprises in three equations (one for price and two for the volatility components).

The REGARCH Option Pricing Model

Derivation of Stochastic Differential Equations

The underlying model is of discrete-time REGARCH type. The processes for price and volatilities can be written as:

$$
\begin{aligned}
dP &= \mu P\, dt + \sigma P\, dX^{(1)} \\
d(\ln\sigma^{(1)}) &= \frac{\kappa_h}{s}(\ln\sigma^{(2)} - \ln\sigma^{(1)})dt + \frac{\delta_h}{\sqrt{s}}dX^{(1)} + \frac{\phi_h}{\sqrt{s}}dX^{(2)} \\
d(\ln\sigma^{(2)}) &= \frac{\kappa_q}{s}(\theta - \frac{1}{2}\ln s - \ln\sigma^{(2)})dt + \frac{\delta_q}{\sqrt{s}}dX^{(1)} + \frac{\phi_q}{\sqrt{s}}dX^{(2)}.
\end{aligned}
\tag{4.29}
$$

In this we have used μ to represent the total real rate, including dividends. The $dX^{(1)}$ and $dX^{(2)}$ are uncorrelated. The coefficients of $dX^{(1)}$ and $dX^{(2)}$ are the 'volatility coefficients' referred to in the spreadsheet.

(The later Appendix 3 of this chapter on Theoretical Concepts contains the original specification of the discrete-time REGARCH model and its conversion to continuous-time stochastic differential equations. It can easily be confirmed that the above system of equations has exactly the same correlation behavior as the system described later.)

This shows that there are only two random factors in the model. However, there are still three state variables required to specify the state of the world (one price and two volatilities).

The main benefit of recasting the discrete-time REGARCH model in continuous time is that it gives us a solid foundation on which to build up an option-pricing model and its associated hedging strategy. Without that foundation we cannot say for certain that the option model is correct, can be used for hedging or can be used for pricing of American options, for example. (This is true even though inevitably a return to a discrete-time model will be necessary for the final number-crunching stage.)

The above stochastic differential equations represent the real variables, the stock price and two volatilities. For the pricing of options we must work in the risk-neutral world.

Since neither of the volatilities is a traded quantity we can expect that the real drifts of both equations will be modified by the market prices of risk associated with the two sources of randomness.

The market price of stock risk is well known to be the quantity

$$\frac{\mu - r}{\sigma^{(1)}}.$$

It is effectively the Sharpe ratio. Note how it is an expression involving the real drift of the asset and its volatility. This is still valid even when that volatility is varying or is itself stochastic. *Since the underlying stock is a traded quantity this market price of risk can be expressed in terms of other quantities already in the model.*

The adjustment to the volatility processes for the market price of stock risk is via the subtraction of a term which is simply the product of the market price of stock risk and the relevant volatility of volatility factor.

For the $\sigma^{(1)}$ process we must subtract

$$\frac{(\mu - r)\delta_h}{\sigma^{(1)}\sqrt{s}}$$

and for the $\sigma^{(2)}$ process we must subtract

$$\frac{(\mu - r)\delta_q}{\sigma^{(1)}\sqrt{s}}.$$

This deals with the market price of stock price risk.

Since the second factor $dX^{(2)}$ is unhedgeable (neither of the volatilities is traded) we cannot express the market price of risk associated with it in terms of other, known or modeled, quantities. It must remain an additional, and separately modeled, quantity. We shall denote it by λ. Again, we must modify the volatility drifts by subtracting terms of the form

$$\frac{\lambda\phi_h}{\sqrt{s}}$$

from the drift of the first volatility and

$$\frac{\lambda\phi_q}{\sqrt{s}}$$

from the second.

Therefore we find that the risk-neutral equivalents of the original stochastic differential equations are

$$\begin{aligned}
dP &= (rP - D)\,dt + \sigma P\,dX^{(1)} \\
d(\ln\sigma^{(1)}) &= \left(\frac{\kappa_h}{s}\left(\ln\sigma^{(2)} - \ln\sigma^{(1)}\right) - \frac{(\mu - r)\delta_h}{\sigma^{(1)}\sqrt{s}} - \lambda\frac{\phi_h}{\sqrt{s}}\right)dt + \frac{\delta_h}{\sqrt{s}}\,dX^{(1)} + \frac{\phi_h}{\sqrt{s}}\,dX^{(2)} \\
d(\ln\sigma^{(2)}) &= \left(\frac{\kappa_q}{s}\left(\theta - \frac{1}{2}\ln s - \ln\sigma^{(2)}\right) - \frac{(\mu - r)\delta_q}{\sigma^{(1)}\sqrt{s}} - \lambda\frac{\phi_q}{\sqrt{s}}\right)dt + \frac{\delta_q}{\sqrt{s}}\,dX^{(1)} + \frac{\phi_q}{\sqrt{s}}\,dX^{(2)}.
\end{aligned} \tag{4.30}$$

These are the random walks that must be simulated for pricing.

The new parameter λ (which may be a function of the other state variables) is the market price of risk associated with the unhedgeable factor $dX^{(2)}$. The term containing μ is also a market price of risk term, but associated with the stock. The term associated with the market price of stock risk is very small compared with the other volatility drift terms. This has been ignored in the spreadsheet.

The option sensitivities or "Greeks" (delta, vega etc.) have their usual meanings. However, it is worth mentioning that all $dX^{(1)}$ risk is totally eliminated by hedging with a short stock position of

$$\Delta_B = \frac{\partial V}{\partial P} + \frac{1}{\sigma^{(1)} P \sqrt{s}} \left(\delta_h \frac{\partial V}{\partial (\ln \sigma^{(1)})} + \delta_q \frac{\partial V}{\partial (\ln \sigma^{(2)})} \right). \tag{4.31}$$

To see this, just calculate the coefficient of $dX^{(1)}$ in the stochastic differential equation for V using Ito's lemma.

From Ito's lemma we have

$$dV = \left(\frac{\partial V}{\partial t} + ... \right) dt + \left(\sigma^{(1)} P \frac{\partial V}{\partial P} + \frac{1}{\sqrt{s}} \left(\delta_h \frac{\partial V}{\partial (\ln \sigma^{(1)})} + \delta_q \frac{\partial V}{\partial (\ln \sigma^{(2)})} \right) \right) dX^{(1)} + (...) dX^{(2)} \tag{4.32}$$

But for hedging purposes we hold a quantity Δ_B of the stock short. This contributes a term

$$-\Delta_B \left(\sigma^{(1)} P \, dX^{(1)} + ... dt \right) \tag{4.33}$$

to the 'hedged' portfolio. Now remember that Δ_B is chosen to eliminate the $dX^{(1)}$ randomness in the dV expression.

This Δ_B may be termed the 'best hedge ratio.' It is calculated in the spreadsheet. An option hedged in such a way has a return in excess of the risk-free rate of

$$\frac{1}{\sqrt{s}} \left(\phi_h \frac{\partial V}{\partial (\ln \sigma^{(1)})} + \phi_q \frac{\partial V}{\partial (\ln \sigma^{(2)})} \right) \left(\lambda \, dt + dX^{(2)} \right) \tag{4.34}$$

The easiest way to derive this expression is to calculate the coefficient of $dX^{(2)}$ in the stochastic differential equation for V using Ito's lemma:

$$dV = \left(\frac{\partial V}{\partial t} + ... \right) dt + (...) dX^{(1)} + \frac{1}{\sqrt{s}} \left(\phi_h \frac{\partial V}{\partial (\ln \sigma^{(1)})} + \phi_q \frac{\partial V}{\partial (\ln \sigma^{(2)})} \right) dX^{(2)} \quad (4.35)$$

The coefficient of $dX^{(2)}$ in this represents the risk that cannot be eliminated. Now simply add to this a deterministic term (i.e. a dt term) with the same coefficient multiplied by the relevant market price of risk.

This expression shows how there is a 'residual risk' that is unhedgeable. The magnitude of this risk is the coefficient of $dX^{(2)}$ and is calculated in the spreadsheet.

Simulations versus partial differential equations

The above stochastic differential equations can be used for the calculation of prices and Greeks via a direct simulation. The same stochastic differential equation foundation can also be used to cast the pricing and hedging problem in terms of a partial differential equation. This would allow other numerical methods to be used, in particular ones which could be extended to the pricing of American options, something which is very difficult to do in a pure simulation framework.

Special case, deterministic volatility

With

$$D = 0, \gamma_h = 0, \ \kappa_q = 0 \text{ and } \gamma_q = 0$$

the system of equations reduces to

$$dP = rP\,dt + \sigma^{(1)} P\,dW^{(1)}, \tag{4.36}$$

$$d(\ln(\sigma^{(1)}) = \frac{\kappa_h}{s}(\ln(\sigma^{(2)}) - \ln(\sigma^{(1)}))dt, \tag{4.37}$$

$$d(\ln(\sigma^{(2)}) = 0.$$

The volatilities are then both deterministic

$$\sigma^{(2)} = \text{constant}$$

$$\ln(\sigma^{(1)}) = \ln(\sigma^{(2)}) + A\exp\left(-\frac{\kappa_h}{s}t\right). \tag{4.38}$$

When volatility is deterministic the Black-Scholes formulae are still valid, provided the volatility is replaced by the square root of the average variance. That is, replace the (constant) volatility in the Black-Scholes formulae with

$$\sqrt{\frac{1}{t}\int_0^t (\sigma^{(1)}(\tau))^2\,d\tau}\,. \tag{4.39}$$

Monte-Carlo Simulation Model

A Monte-Carlo simulation model based on the above analysis was programmed in MS-Excel with spreadsheet inputs and outputs described below.

Required Inputs

Asset Today	100
Volatility_1 Today	20.00%
Volatility_2 Today	20.00%

Asset Today is the underlying stock price at time of pricing. **Volatility_1 Today** and **Volatility_2 Today** are the values of the two volatility variables, output from the REGARCH analysis. These three variables must be updated at each new pricing.

Market Price of Volatility_1 Risk	0

Market Price of Volatility_1 Risk is a parameter, and should not need changing.

Market Price of Volatility_2 Risk	0

Market Price of Volatility_2 Risk is a parameter, and should not need changing.

Interest rate	0.05

Interest rate represents the interest rate applying over the option's life.

Number of timesteps	100
Number of paths	200

Number of timesteps and **Number of paths** govern the accuracy of the end result. If the number of timesteps is too few for the calculation to converge, an error message will be output and the function will be exited.

REGARCH Model Parameters

κh	0.5163
ϕh	0.0566
δh	-0.0012
κq	0.0249
θq	-4.4382
ϕq	0.0058
δq	-0.013

The above are the process parameters, output from the REGARCH model estimation procedure. They should remain fixed for each stock, but will vary from stock to stock.

Dividends
(Continuous yield)

Yield	0.0%

Dividends (Discrete amount)

Time	Amount
0.25	1
0.75	2
2	0

The spreadsheet allows a very general dividend structure in which discrete dividends of any size can be scheduled for payment at any point during the life of the option(s) to be modelled.

	A	B	C	D	E	F	G
Type	C	C	C	C	C	C	C
Strike	70	80	90	100	110	120	130
Expiration	1	1	1	1	1	1	1

Seven option series have been included (A-G), but this can easily be extended.

Output

Theoretical values for the option, the Delta, Gamma and Theta are output for each option series.

Option Value	34.089	25.752	18.511	12.853	8.592	5.509	3.454
Delta	0.953	0.884	0.766	0.618	0.488	0.352	0.248
Gamma	0.003	0.005	0.014	0.013	0.010	0.014	0.014
Theta	-4.091	-7.044	-6.780	-7.636	-6.400	-4.977	-5.807
Vega 1	0.145	0.282	0.489	0.628	0.794	0.681	0.744
Vega 2	3.683	7.362	11.927	14.290	15.464	14.628	12.237

These option values are then reinterpreted in terms of implied volatilities.

Implied Volatility	27.4%	26.7%	26.2%	26.4%	26.4%	26.3%	26.3%

These implied volatilities are then fed back into the Black-Scholes formulae to give reference values for the Greeks.

Delta	0.947	0.876	0.765	0.626	0.484	0.355	0.250
Gamma	0.004	0.008	0.012	0.014	0.015	0.014	0.012
Theta	-4.502	-5.826	-6.926	-7.482	-7.258	-6.398	-5.252
Vega	10.722	20.455	30.703	37.885	39.863	37.235	31.751

The model also calculated the optimal hedge ratio and the residual risk:

Best hedge ratio	0.952	0.882	0.762	0.613	0.482	0.348	0.244
Residual risk	0.317	0.632	1.031	1.244	1.372	1.282	1.111

The implied volatilities are plotted against strike prices.

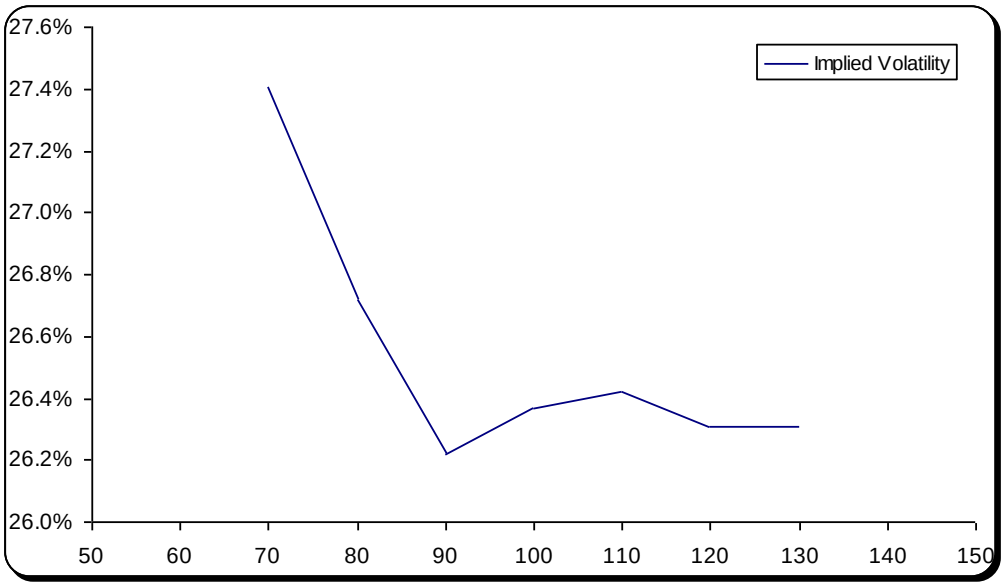

Figure 1 Implied Volatilities from REGARCH Option Model

Model Accuracy and Performance

Comparison with Black-Scholes

There are very few exact, known solutions for complex option-pricing models against which to test the Monte Carlo code. One of the few exact solutions is the basic Black-Scholes formula but applied in the case of deterministic (i.e. not stochastic) volatility. For testing purposes we have compared the model output with explicit formulae for option values under assumptions of constant volatility and deterministic volatility. In all cases, including with dividends proportional to stock price, the results converged to the correct result and were accurate to within the expected tolerances.

In comparing the Greek parameters of the REGARCH vs Black-Scholes model, the major difference is likely to be seen in the option Vegas. In the REGARCH model there are two Vega parameters, the first quantifying sensitivity to changes

in the short-term volatility (the h_t process) and the second parameter measuring sensitivity to changes in long-term volatility (the q_t process). In general, Vega 1 will be far lower than the standard Black-Scholes Vega because of the rapid speed of mean-reversion in the transitory volatility process. Vega 2, the sensitivity to long term volatility, will typically be of the same order of magnitude and the standard Black-Scholes Vega, because the speed of mean reversion in the long-term volatility process is comparatively low. It follows that Vega 1 will be negatively correlated to the mean reversion parameter kappa-h, while Vega 2 will be negatively correlated to kappa-q. For very small values of kappa-q, the REGARCH Vega 2 will be very close to the standard Black-Scholes Vega.

To demonstrate this relationship we examined the Vegas of a range of 1-year call options with strikes in the range of 70 to 130 on an asset having the same REGARCH parameters as those estimated for the KOSPI index. We examined the behavior of the Vega 1 and Vega 2 Greeks as kappa-h and kappa-q were varied in a range from 0 to 0.01. The results shown in the chart below clearly demonstrate the inverse relationship between kappa-q and Vega 2 and show that, for value of kappa-q around 0.001, the mean reversion effect is negligibly small - at that point Vega 2 becomes almost indistinguishable from the standard Black-Scholes Vega. By contrast, options on a stock process having a mean reversion parameter for the long-run volatility process as high as 0.01 are relatively insensitive to changes in long term volatility. In this case Vega 2 is as low as 42% of the standard Black-Scholes Vega.

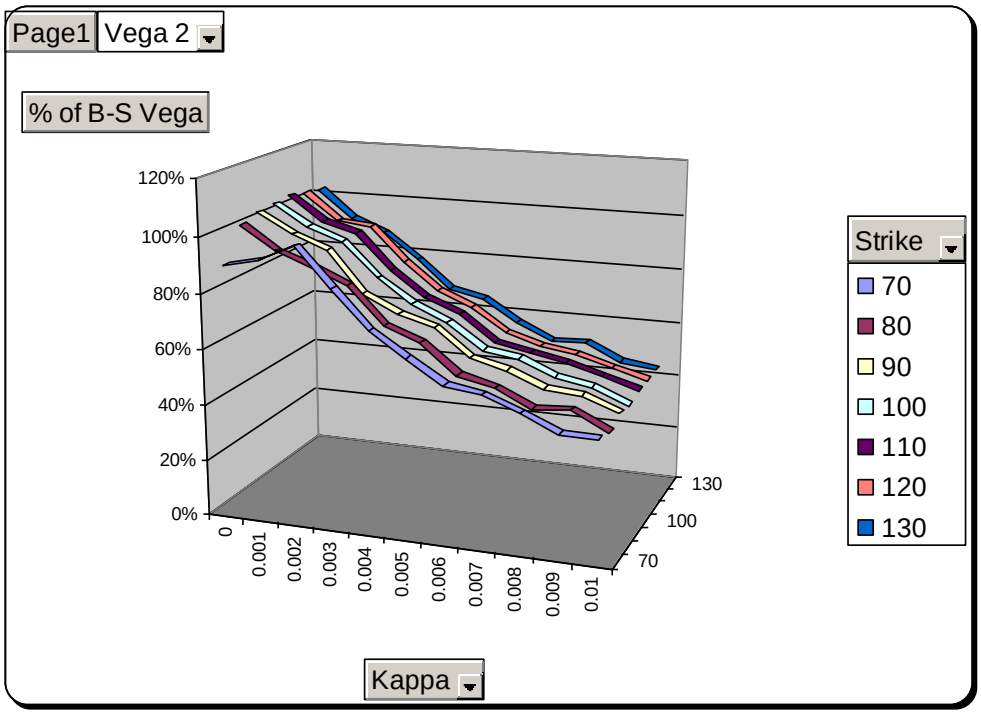

Figure 2 Option Vegas Computed Using REGARCH Model

Accuracy Tests

Given a Monte Carlo estimator $\hat{P}_N$ of the average price of an option based on N simulations (number of paths), then from the Central Limit Theorem the standard deviation of the estimate approaches $\sigma/\sqrt{N}$ for large N.

The variance σ^2 of the random variate P whose expectation we are trying to estimate is unknown. However we can use the sample standard deviation $\hat{\sigma}_N$ as

an unbiased and efficient estimator of σ. This leads to the definition of the standard error of $\hat{P}_N$:

$$SE_N = \frac{\hat{\sigma}_N}{\sqrt{N}}$$

The % accuracy can then be expressed as $SE_N / \hat{P}_N$.

We conducted accuracy tests for 6-month call options at strikes between 70 and 130 (100 = ATM) using the REGARCH model parameters estimated for the KOSPI index. Average prices, sample standard deviations and standard errors were estimated for simulations of between 500 and 10,000 paths (in increments of 500), with between 50-250 time-steps (in increments of 50 time steps). The results are summarized in the table and chart below.

	50	100	150	200	250
500	6.75%	6.96%	7.19%	7.20%	7.16%
1000	5.21%	4.91%	5.07%	5.08%	5.04%
1500	4.18%	4.04%	4.16%	4.11%	4.12%
2000	3.62%	3.45%	3.61%	3.57%	3.54%
2500	3.19%	3.10%	3.21%	3.19%	3.15%
3000	2.83%	2.86%	2.95%	2.91%	2.90%
3500	2.64%	2.65%	2.73%	2.67%	2.71%
4000	2.45%	2.52%	2.54%	2.49%	2.49%
4500	2.38%	2.31%	2.37%	2.34%	2.39%
5000	2.21%	2.23%	2.25%	2.22%	2.26%
5500	2.13%	2.13%	2.17%	2.17%	2.14%
6000	2.01%	2.05%	2.08%	2.04%	
6500	1.95%	1.95%	1.99%	1.96%	
7000	1.94%	1.88%	1.90%	1.86%	
7500	1.86%	1.82%	1.87%	1.83%	
8000	1.77%	1.80%	1.78%	1.76%	
8500	1.67%	1.74%	1.74%	1.71%	
9000	1.69%	1.65%	1.66%	1.68%	
9500	1.63%	1.61%	1.63%	1.61%	
10000	1.56%	1.60%	1.61%	1.61%	

Table 1 % Standard Error for ATM Options

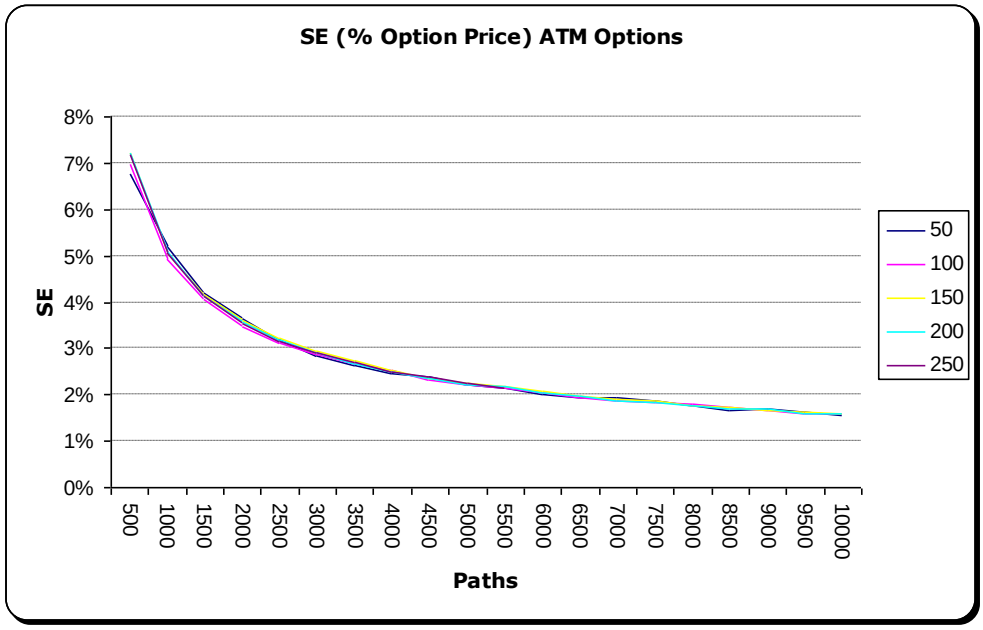

Figure 3 % Standard Error of ATM (100 Strike) Call Options

The analysis confirms the theoretical result that the standard error declines with the square-root of the number of simulations. For a sizeable number of simulations (4,000 or more) the number of time steps used makes relatively little difference to the accuracy level.

The results indicate that, for ATM options, an acceptable level of accuracy (say, around 2.5%) can be achieved with 4,000 simulations.

However, the same does not hold true for OTM options. As the table and chart below indicate, in addition to the number of simulations, accuracy levels depend on the money-ness of the options. Due to the extreme kurtosis of the sample price distribution, the percentage standard errors of OTM option price estimates

are uniformly higher than for ATM or ITM options. To take a specific example, the percentage standard error of a call option estimate based on 4,000 simulations is, at 6.86%, some 2.7 time greater than that for an ATM option, and around 6.5 times larger than for an option struck at 30 points ITM. The implication is that a far larger number of simulations will be required to achieve a desired degree of accuracy in the estimation of deep OTM options, compared with ATM or ITM options.

	Strike						
Paths	**70**	**80**	**90**	**100**	**110**	**120**	**130**
500	2.94%	3.85%	5.15%	6.96%	9.55%	13.50%	19.68%
1000	2.08%	2.72%	3.64%	4.91%	6.74%	9.54%	13.95%
1500	1.70%	2.23%	2.99%	4.04%	5.53%	7.78%	11.26%
2000	1.46%	1.92%	2.56%	3.45%	4.74%	6.71%	9.80%
2500	1.31%	1.71%	2.29%	3.10%	4.24%	5.96%	8.67%
3000	1.21%	1.58%	2.11%	2.86%	3.92%	5.50%	7.87%
3500	1.12%	1.46%	1.96%	2.65%	3.64%	5.10%	7.32%
4000	1.06%	1.38%	1.85%	2.52%	3.47%	4.86%	6.86%
4500	0.98%	1.27%	1.70%	2.31%	3.16%	4.44%	6.40%
5000	0.93%	1.22%	1.64%	2.23%	3.08%	4.34%	6.17%
5500	0.89%	1.17%	1.57%	2.13%	2.91%	4.07%	5.83%
6000	0.87%	1.13%	1.51%	2.05%	2.79%	3.87%	5.49%
6500	0.82%	1.08%	1.44%	1.95%	2.67%	3.76%	5.47%
7000	0.78%	1.03%	1.38%	1.88%	2.58%	3.58%	5.08%
7500	0.76%	1.00%	1.35%	1.82%	2.49%	3.48%	4.94%
8000	0.75%	0.98%	1.32%	1.80%	2.46%	3.42%	4.87%
8500	0.72%	0.95%	1.28%	1.74%	2.39%	3.33%	4.68%
9000	0.70%	0.92%	1.22%	1.65%	2.28%	3.19%	4.58%
9500	0.67%	0.88%	1.18%	1.61%	2.22%	3.10%	4.45%
10000	0.67%	0.88%	1.18%	1.60%	2.20%	3.05%	4.26%

Table 2 % Standard Errors for Call Option Price Estimates (100 time steps)

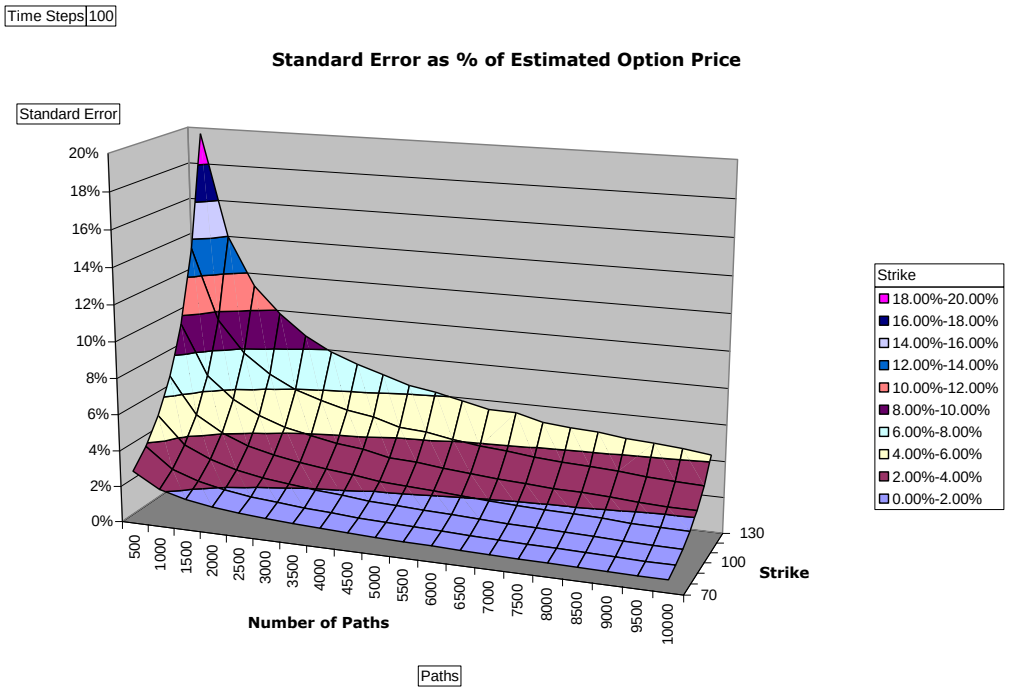

Figure 4 % Standard Errors of 6-M Call Options.

Performance Testing

Performance tests were carried out to analyze the time taken by the model to price a series of seven options (strikes from 70 to 130, 1-year expiration), calculate Greek parameters, implied volatilities and, for comparison purposes, the corresponding prices and Greeks from the Black-Scholes model. The results are summarized in the table below, which shows the time taken (in seconds) to complete the computations using a number of simulations (in the range from 500 to 10,000) and time-steps (from 100 to 500). In practice, the number of time steps makes relatively little difference to the accuracy levels, so the most relevant results are those in the first column. The results show that the total computation time is a simple linear function of the number of paths and time steps of the form:

T(secs) = 0.0006 x Time-Steps x Paths

	Time Steps				
Paths	**100**	**200**	**300**	**400**	**500**
500	33	62	91	121	150
1000	63	124	182	242	301
1500	95	184	274	362	451
2000	127	246	364	482	653
2500	158	307	456	883	749
3000	216	403	544	721	898
3500	221	428	634	841	1,047
4000	253	489	725	962	1,198
4500	285	551	815	1,083	1,347
5000	337	618	912	1,206	1,511
5500	369	679	1,000	1,327	1,693
6000	387	823	1,160	1,453	1,816
6500	412	803	1,184	1,573	1,953
7000	444	859	1,310	1,689	2,107
7500	487	938	1,368	1,825	2,309
8000	522	995	1,472	1,947	2,425
8500	538	1,040	1,541	2,044	2,546
9000	571	1,101	1,631	2,166	2,695
9500	602	1,165	1,735	2,291	2,855
10000	636	1,234	1,818	2,413	3,078

Table 3 Computation Times (in seconds)

The Impact of the Market Price of Volatility Risk

The form of the diffusion equations leads us to expect that, for both short- and long-term volatility processes, the market price of volatility risk (MKPRVR) will be negatively correlated with the overall level of volatility. Consequently the principal effect of increasing MKPRVR will be to reduce the average level of the implied volatilities of the options. To demonstrate this effect we used the REGARCH model to estimate implied volatilities for a series of 6-month put options on the South Korean KOSPI index (using the REGARCH model

parameters estimated from Stage 1 of the project and simulating 4,000 paths with 50 time-steps). The volatility smiles were estimated for levels of MKPRVR in the range from 0% to 100%, in increments of 10%.

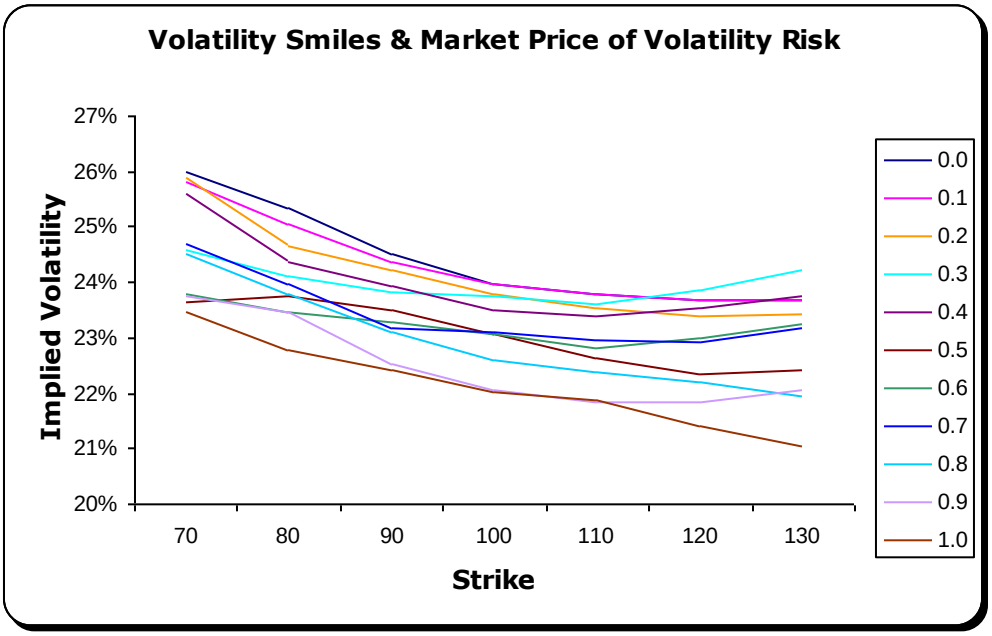

Figure 5 Implied Volatilities of 6-M Put Options on the South Korean KOSPI Index

The effect of the interaction between the two market price of volatility risk factors on average implied volatilities is shown in the chart below. The impact of increasing levels of MKPRVR is to lower the average level of implied volatility across all strikes. For a given level of MKPRVR, the effect of an increase in the second MKPRVR factor of 10% is to reduce the average level of implied volatility (across all strikes) by an average of between -0.1% to -0.4%.

	70	80	90	100	110	120	130
0.0	26.0%	25.4%	24.5%	24.0%	23.8%	23.7%	23.7%
0.1	25.8%	25.1%	24.3%	24.0%	23.8%	23.7%	23.7%
0.2	25.9%	24.7%	24.2%	23.8%	23.5%	23.4%	23.4%
0.3	24.6%	24.1%	23.8%	23.8%	23.6%	23.8%	24.2%
0.4	25.6%	24.4%	23.9%	23.5%	23.4%	23.6%	23.8%
0.5	23.7%	23.8%	23.5%	23.1%	22.6%	22.4%	22.4%
0.6	23.8%	23.5%	23.3%	23.1%	22.8%	23.0%	23.2%
0.7	24.7%	24.0%	23.2%	23.1%	23.0%	22.9%	23.2%
0.8	24.5%	23.8%	23.1%	22.6%	22.4%	22.2%	22.0%
0.9	23.8%	23.5%	22.5%	22.1%	21.8%	21.8%	22.1%
1.0	23.5%	22.8%	22.4%	22.0%	21.9%	21.4%	21.0%

Table 4 Average Implied Volatilities of 6-M Put Options on the KOSPI Index by MKPRV

Conclusion

We have shown how the theoretical framework developed by Alizadeh, Brandt and Diebold (2002) can be extended to provide the basis for a new methodology for option pricing, one which takes account of important stylized facts about volatility that have been confirmed repeatedly by empirical research. These characteristics include the stochastic nature of volatility, asymmetry, interaction with the underlying returns process and the decomposition of its behavior into two components, a long term stochastic mean process superimposed upon which is a mean-reverting transient process. A discretized version of the model, adaptable for use in Monte-Carlo simulation, appears to provide a viable approach to generating option prices which incorporate these effects.

Appendix 1 – Theoretical Concepts

Unbiased volatility forecasts for REGARCH models

At time *t*, ARCH models provide h_{t+1} and q_{t+1}. We may want to predict $\ln(h_{t+n})$, h_{t+n} and h_{t+n}^2 for horizons $n > 1$.

Let us introduce the notation

$$\varepsilon_t = \phi_h X_t^D + \delta_h R_t / h_t$$

$$\eta_t = \phi_q X_t^D + \delta_q R_t / h_t$$

for the residuals in these equations. Also, let

$$H_t = \ln(h_t) - \theta$$

$$Q_t = \ln(q_t) - \theta .$$

Then our model for the two-factor volatility process can be written as follows:

$$H_t - H_{t-1} = \kappa_h (Q_{t-1} - H_{t-1}) + \varepsilon_{t-1}$$

$$Q_t - Q_{t-1} = -\kappa_q Q_{t-1} + \eta_{t-1}.$$

It is convenient to introduce the autoregressive parameters

$$\alpha = 1 - \kappa_h$$

$$\beta = 1 - \kappa_q$$

Forecasts are obtained by writing future log-volatilities as the sum of variables known at the present time plus a linear combination of unknown residuals. Tedious algebra leads to :

$$H_{t+n} = \alpha^{n-1} H_{t+1} + (1-\alpha)\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) Q_{t+1} + \omega_{t,n}$$

for $n \geq 2$, with $\omega_{t,n}$ a zero-mean forecast error defined by :

$$\omega_{t,2} = \varepsilon_{t+1}$$

$$\omega_{t,3} = \varepsilon_{t+2} + \alpha\varepsilon_{t+1} + (1-\alpha)\eta_{t+1}$$

…..

$$\omega_{t,n} = \sum_{i=1}^{n-1} a_i \varepsilon_{t+n-i} + b_i \eta_{t+n-i}$$

and

$$a_1 = 1,\ a_2 = \alpha,\ .....\ a_i = \alpha^{i-1}$$

$$b_1 = 0,\ b_2 = 1-\alpha,\ ....\ b_i = (1-\alpha)(\alpha^{i-1} - \beta^{i-1})/(\alpha - \beta),\ i \geq 2.$$

Forecasts

Setting the forecast error equal to zero gives the forecast of $\ln(h_{t+n})$ as

$$f_{t,n} = \theta + \alpha^{n-1} H_{t+1} + (1-\alpha)\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) Q_{t+1}$$

To predict h_{t+n} or h_{t+n}^2 requires the variance of the forecast error $\omega_{t,n} = \ln(h_{t+n}) - f_{t,n}$. This is given by

$$\text{var}(\omega_{t,n}) = \sum_{i=1}^{n-1} a_i^2(\phi_h^2 + \delta_h^2) + 2a_i b_i(\phi_h\phi_q + \delta_h\delta_q) + b_i^2(\phi_q^2 + \delta_q^2).$$

Simplifying this expression is not easy. It may be needed for several values of *n* and can then be calculated recursively as

$$\text{var}(\omega_{t,n}) = \text{var}(\omega_{t,n-1}) + a_{n-1}^2(\phi_h^2 + \delta_h^2) + 2a_{n-1}b_{n-1}(\phi_h\phi_q + \delta_h\delta_q) + b_{n-1}^2(\phi_q^2 + \delta_q^2)$$

for $n \geq 3$.

Assuming the forecast errors are normally distributed, the unbiased forecast of h_{t+n} is

$$\hat{h}_{t+n} = \exp(f_{t,n} + 0.5\,\text{var}(\omega_{t,n}))$$

and the unbiased forecast of h_{t+n}^2 is

$$g_{t,n} = \exp(2f_{t,n} + 2\,\text{var}(\omega_{t,n})).$$

An unbiased forecast of the average variance

$$\frac{1}{N}\sum_{j=1}^{N} h_{t+j}^2$$

is given by averaging the variance forecasts, to give

$$\frac{1}{N}\sum_{j=1}^{N} g_{t,j}\,.$$

The logarithm of the variable q follows an AR(1) process that we assume is Gaussian.

Let

$$Q_t = \ln(q_t) - \theta\,.$$

Then

$$Q_{t+1} = \beta Q_t + \eta_t$$

with

$$\eta_t = \phi_q X_t^D + \delta_q R_t / h_t\,.$$

Also let $\psi = \mathrm{var}(X_t^D)$ and $\lambda = \mathrm{var}(R_t / h_t)$. Then

$$\sigma_\eta^2 = \mathrm{var}(\eta_t) = \phi_q^2 \psi + \delta_q^2 \lambda.$$

At time t we know Q_{t+1} and from properties of an AR(1) we have conditional distributions

$$Q_{t+n} | Q_{t+1} \sim N(\beta^{n-1} Q_{t+1}, \frac{1-\beta^{2(n-1)}}{1-\beta^2}\sigma_\eta^2)$$

and hence the optimal forecast of q_{t+n} is

$$E[q_{t+n}|q_{t+1}] = \exp(\theta + \beta^{n-1}(\ln(q_{t+1}) - \theta) + 0.5\frac{1-\beta^{2(n-1)}}{1-\beta^2}\sigma_\eta^2).$$

Mean square errors for forecasts

The mean square error (MSE) is the expectation of the squared forecast error. The standard deviation of the forecast error is the square root of the MSE.

$\ln(h_{t+n}) = f_{t,n} + \omega_{t,n}$ so the MSE is then $\text{var}(\omega_{t,n})$ which is given by a long formula in the previous section.

Let $V_{t,n} = \text{var}(\omega_{t,n})$ to make the notation more compact.

For h_{t+n}, the forecast is $\hat{h}_{t+n} = \exp(f_{t,n} + 0.5V_{t,n})$. As h_{t+n} has a lognormal distribution we can derive

$$MSE(h_{t+n}) = E[(h_{t+n} - \hat{h}_{t+n})^2] = e^{2f_{t,n}+V_{t,n}}\left(e^{V_{t,n}} - 1\right).$$

For h_{t+n}^2, the forecast is $g_{t,n} = \exp(2f_{t,n} + 2\text{var}(\omega_{t,n}))$ and

$$MSE(h^2{}_{t+n}) = e^{4f_{t,n}+4V_{t,n}}\left(e^{4V_{t,n}} - 1\right)$$

MSE for the average variance

The average variance is $A = \frac{1}{N}\sum_{j=1}^{N} h_{t+j}^2$ and it is predicted by $\hat{A} = \frac{1}{N}\sum_{j=1}^{N} g_{t,j}$

with $g_{t,j} = \exp(2f_{t,j} + 2V_{t,j})$. Assume $N \geq 2$.

The mean square error of A is

$$MSE(A) = E[(A-\hat{A})^2] = \frac{1}{N^2} E[\sum_{j=1}^{N} (h_{t+j}^2 - g_{t,j})^2]$$

$$= \frac{1}{N^2}\sum_{j=1}^{N} MSE(h_{t+j}^2) + \frac{2}{N^2}\sum_{j=1}^{N-1}\sum_{k=j+1}^{N} E[(h_{t+j}^2 - g_{t,j})(h_{t+k}^2 - g_{t,k})]$$

$$= \frac{1}{N^2}\sum_{j=1}^{N} e^{4f_{t,j}+4V_{t,j}}(e^{4V_{t,j}} - 1) + \frac{2}{N^2}\sum_{j=1}^{N-1}\sum_{k=j+1}^{N} E[h_{t+j}^2 h_{t+k}^2] - g_{t,j}g_{t,k}.$$

This can be evaluated once we know $E[h_{t+j}^2 h_{t+k}^2]$ for $j < k$

as

$$\ln(h_{t+j}^2 h_{t+k}^2) = 2[f_{t,j} + f_{t,k} + \omega_{t,j} + \omega_{t,k}]$$

has an assumed Normal distribution, the required expectation is

$$E[h_{t+j}^2 h_{t+k}^2] = \exp(2f_{t,j} + 2f_{t,k} + 2W_{t,j,k})$$

with

$$W_{t,j,k} = \mathrm{var}(\omega_{t,j} + \omega_{t,k}).$$

If j is 1 then $W_{t,1,k} = V_{t,k}$.

Now assume $j \geq 2$ and $k, N \geq 3$. To find the terms *W*, adapt one of the equations in the previous analysis to give

$$\omega_{t,j} = \sum_{i=1}^{j-1} a_{j-i}\varepsilon_{t+i} + b_{j-i}\eta_{t+i}.$$

Then

$$\omega_{t,j} + \omega_{t,k} = \sum_{i=1}^{k-1} c_{i,j,k}\varepsilon_{t+i} + d_{i,j,k}\eta_{t+i}$$

with

$$\begin{aligned} c_{i,j,k} &= a_{j-i} + a_{k-i} \quad \text{if} \quad 1 \leq i \leq j-1, \\ &= a_{k-i} \qquad\quad \text{if} \quad j \leq i \leq k-1 \end{aligned}$$

and

$$\begin{aligned} d_{i,j,k} &= b_{j-i} + b_{k-i} \quad \text{if} \quad 1 \leq i \leq j-1, \\ &= b_{k-i} \qquad\quad \text{if} \quad j \leq i \leq k-1 \end{aligned}$$

These equations allow the terms *W* to be calculated from:

$$W_{t,j,k} = \sum_{i=1}^{k-1} c_{i,j,k}^2(\phi_h^2\psi + \delta_h^2\lambda) + 2c_{i,j,k}d_{i,j,k}(\phi_h\phi_q\psi + \delta_h\delta_q\lambda) + d_{i,j,k}^2(\phi_q^2\psi + \delta_q^2\lambda).$$

Monte-Carlo Simulation

The value of an option on a stock S is given by the equation $P = e^{-rt}E[payoff(S)]$, providing the expectation is with respect to a risk-neutral

random walk. Hence we can estimate the value of the option using Monte Carlo simulation as follows:

Simulate the risk neutral random walk, evolving the stock process until option expiration.

1. For this realization calculate the option payoff.
2. Repeat the above procedure for many simulations.
3. Calculate the average payoff over all realizations.
4. Take the present value of the average option payoff.

The continuous time risk-neutral stock process is simulated using a discrete procedure known as the **Euler method** with the equation:

$$\delta S = rS\delta t + \sigma S\sqrt{\delta t}\phi$$

where ϕ is drawn from a standardized Normal distribution.

The expression for the stock price process is then:

$$S(t+\delta t) = S(t)e^{([r-\sigma^2/2]\delta t+\sigma\sqrt{\delta t}\phi)}$$

The error in the estimated option price using the Monte Carlo procedure with N realizations is:

$$O(\max\left[\delta t, \frac{1}{\sqrt{N}}\right]$$

Normally distributed random numbers are generated using the **Box-Muller method**. The procedure is to take two uniform random variates x1 and x2 and combine them to give two Normally distributed variates y1 and y2 as follows:

$$y_1 = \sqrt{-2\ln x_1}\cos(2\pi x_2) \quad \text{and} \quad y_2 = \sqrt{-2\ln x_1}\sin(2\pi x_2).$$

In the REGARCH framework we need to extend the Euler method to three dimensions with the formulation:

$$S_i(t+\delta t) = S_i(t)e^{([r-\sigma_i{}^2/2]\delta t+\sigma_i\sqrt{\delta t}\phi_i)}$$

The issue here is that the ϕ_i (i = 1, . . , 3) are correlated random variables. We generate these using **Cholesky factorization**. The procedure is as follows.

Suppose we have d uncorrelated Normally distributed variables $\varepsilon_1, \ldots, \varepsilon_d$. We can then generate correlated variables with the transformation: $\boldsymbol{\varphi} = \mathbf{M}\boldsymbol{\varepsilon}$.

The matrix **A** must satisfy $\mathbf{M}\mathbf{M}^{\mathrm{T}} = \boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma}$ is the correlation matrix.

It is easy to show that $E[\boldsymbol{\varphi}\boldsymbol{\varphi}^T] = \mathbf{M}E[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{\mathrm{T}}]\mathbf{M}^{\mathrm{T}} = \mathbf{M}\mathbf{M}^{\mathrm{T}} = \boldsymbol{\Sigma}$ as required. The decomposition of the correlation matrix into the product of two matrices is not unique. Cholesky factorization is one procedure for doing this which produces a matrix **M** that is lower triangular. The procedure is illustrated below.

$$\Sigma = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{11}a_{21} & a_{11}a_{31} \\ a_{11} & a_{21}^2 + a_{22}^2 & a_{21}a_{31} + a_{32}a_{22} \\ a_{11}a_{31}a_{21}a_{31} + a_{32}a_{22} & a_{11}^2 + a_{22}^2 + a_{33}^2 \end{bmatrix}$$

$$a_{ii} = [s_{ii} - \sum_{k=1}^{i-1} a_{ik}^2]^{1/2}$$

$$a_{ij} = [s_{ij} - \sum_{k=1}^{i-1} a_{ik} a_{jk}] / a_{ii} \quad \text{i} = 1, ..., \text{I}, \ \text{j} = \text{i} + 1, \text{i} + 2, ..., \text{I}$$