Lab: ARMA(1, 1) Process

In this lab we consider an ARMA(1, 1) process of the form:

$$y_t = a_1 y_{t-1} + e_t + b_1 e_{t-1}$$

Where ε_t is a white noise process, mean zero and variance σ^2.

1. Use the ARMA11 worksheet to generate 20 observations from an ARMA(1, 1) process with parameters $a_1 = 0.5$, $\beta_1 = 0.5$. Examine several instances of the process on the chart provided.

2. Use the Yule-Walker equations to derive the autocovariance terms $\gamma_s = E(y_t, y_{t-s})$. Hence derive the ACF terms ρ_t for t = 1, 2, . . . ,20

3. Use the ACF function to compute the sample autocorrelations. Compare the theoretical and sample ACF's on the correlogram provided.

   The ACF function has two parameters ACF(Y, L) where Y is the range of time series observations and L is the lag parameter. It returns the result γ_l.

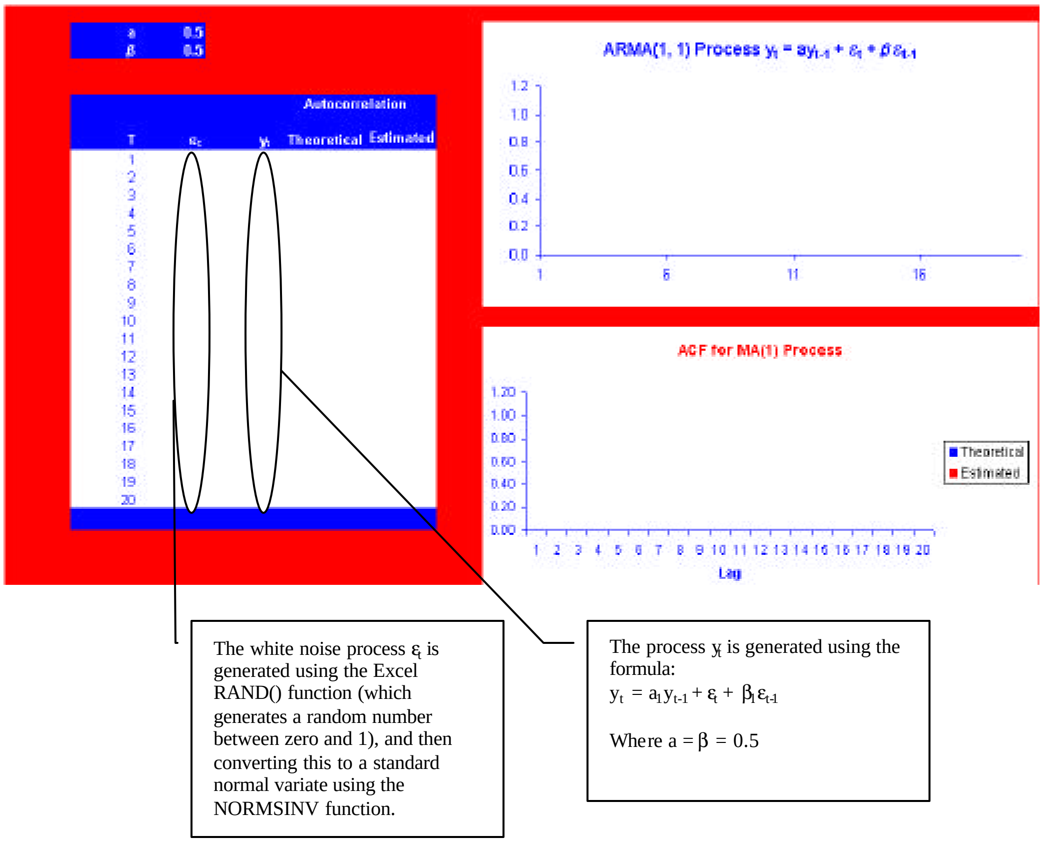
4. Use the PACF function to compute the theoretical and estimated partial autocorrelations and compare them on the correlogram provided.

   The PACF has two parameters PACF(ACF, L) where ACF is the range of actual (or theoretical) autocorrelations and L is the lag parameter. The function returns the result ϕ_{LL}, the partial autocorrelation coefficient at lag L.

5. Experiment with different values of the process coefficients and examine the effects on the form of the ACF and PACF.

Solution: ARMA(1, 1) Process

1. The layout of the lab worksheet ARMA11 is shown below.

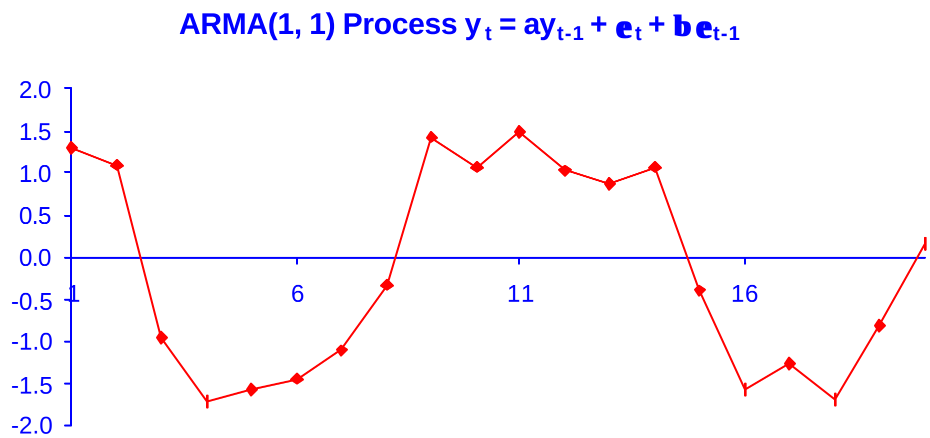


Enter the formula for the first white noise term ε_1 in cell C8 as NORMSINV(RAND()). Copy this formula down into the remaining cells in the column, corresponding to time periods 2 through 20.
In cell D8, enter set the formula for the first process observation y_1 = D8. Then, in cell D9, enter the Excel formula for the second term as follows:

= a*D8 + C9 + b*C8

Copy this formula down into the remaining cells in this column. The time series chart is drawn automatically.
By pressing the function key F9, you can generate different instances of the process. Try various values of the parameters a and β (both positive and negative).

A typical example is shown below.



2. We set up the Yule-Walker equations as follows:

$$\gamma_0 = E(y_t\, y_t) = a_1 E(y_{t-1}\, y_t) + E(\varepsilon_t\, y_t) + \beta_1 E(\varepsilon_{t-1}\, y_t)$$
$$= a_1\gamma_1 + \sigma^2 + \beta_1 E[\varepsilon_{t-1}(a_1 y_{t-1} + \varepsilon_t + \beta_1\varepsilon_{t-1})]$$
$$= a_1\gamma_1 + \sigma^2 + \beta_1(a_1 + \beta_1)\sigma^2$$

$$\gamma_1 = E(y_t\, y_{t-1}) = a_1 E(y_{t-1}\, y_{t-1}) + E(\varepsilon_t\, y_{t-1}) + \beta_1 E(\varepsilon_{t-1}\, y_{t-1})$$
$$= a_1\gamma_0 + \beta_1\sigma^2$$

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$$\gamma_s = E(y_t\, y_{t-s}) = a_1 E(y_{t-1}\, y_{t-s}) + E(\varepsilon_t\, y_{t-}) + \beta_1 E(\varepsilon_{t-1}\, y_{t-s})$$
$$= a_1\gamma_{s-1}$$

Solving these equations simultaneously for γ_0 and γ_1 yields:

$$g_0 = \frac{1+b_1^2+2a_1b_1}{(1-a_1^2)} s^2 \quad \text{and} \quad g_1 = \frac{(1+a_1b_1)(a_1+b_1)}{(1-a_1^2)} s^2$$

Hence, $r_1 = \frac{(1+a_1b_1)(a_1+b_1)}{(1+b_1^2+2a_1b_1)}$ and $\rho_s = a_1\rho_{s-1}$ for s > 1

3. With parameters $a_1 = 0.5$, $\beta_1 = 0.5$ the theoretical ACF is as follows:

$\rho_1 = (1 + 0.5^2)(0.5 + 0.5) / (1 + 0.5^2 + 2(0.5^2)) = 0.714$

In the worksheet the formula is entered into cell E8 as follows:

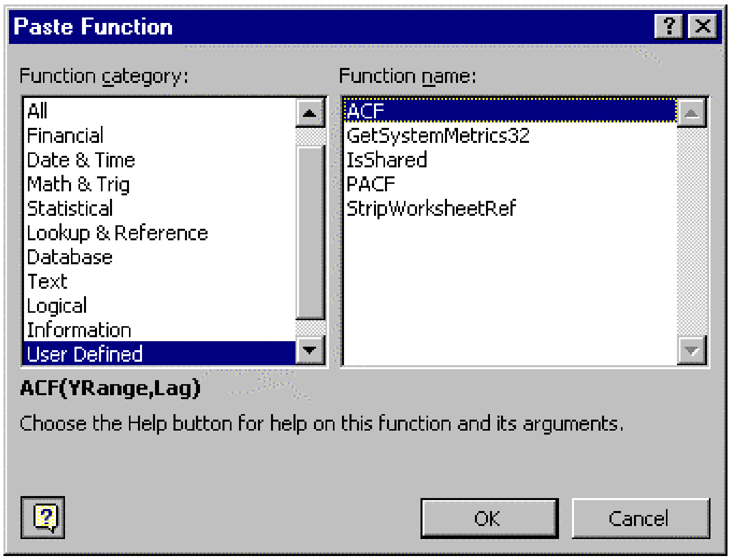
=(1+a*b)*(a+b)/(1+b^2+ 2*a*b)

Subsequent autocorrelations are calculated using the relationship $\rho_s = a_1\rho_{s-1}$.

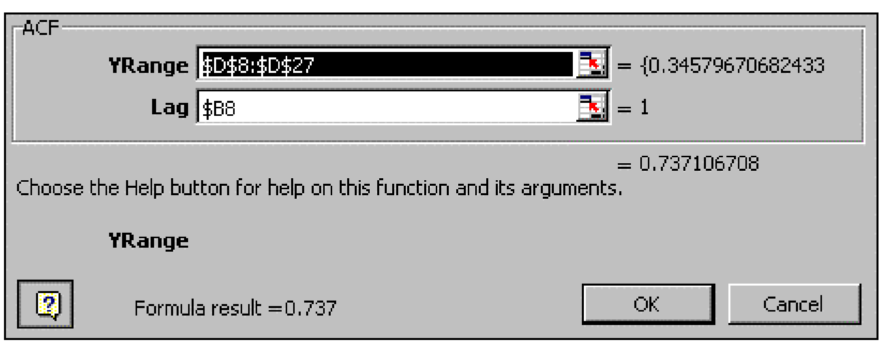
Hence, $\rho_2 = 0.5\ \rho_1 = 0.357$

The formula is entered in cell C9 as: =a*E8. This formula is copied down into the remaining cells in the column.

The sample autocorrelations are calculated using the ACF function. Click the Function Wizard from the menubar and select User Defined functions, ACF as shown in the screenshot below.

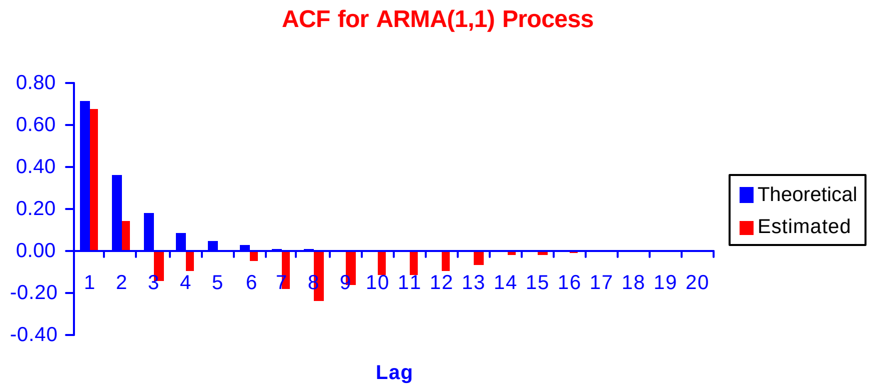


The parameters used in the function are the range of values y_t in cells D8 to D27, and the time lag T in the corresponding cells B8 to B27. The first value is calculated in cell F8, as shown below.



The formula in cell F8 should be copied down to the remaining cells in the column to calculate the autocorrelations for T = 2 to 20.

The ACF chart is automatically updated, showing both the theoretical and sample ACF function values. An example is shown below.



4. The theoretical partial autocorrelation coefficients are given by:

$\phi_{11} = \rho_1$, $\phi_{22} = (\rho_2 - \rho_1^2) / (1 - \rho_1^2)$

and

$$f_{ss} = \frac{r_s - \sum_{j=1}^{s-1} f_{s-1} r_{s-j}}{1 - \sum_{j=1}^{s-1} f_{s-1} r_j}$$

$$f_{sj} = f_{s-1,j} - f_{ss} f_{s-1,s-j} \qquad j = 1, 2, \ldots s-1$$

Hence, $\phi_{11} = \rho_1 = 0.714$,
and $\phi_{22} = (0.357 - 0.714^2) / (1 - 0.714^2) = -0.313$

Thereafter the partial autocorrelations are derived iteratively:

$\phi_{21} = \phi_{11} - \phi_{22}\,\phi_{11} = 0.714 - (-0.313)*0.714 = 0.9375$

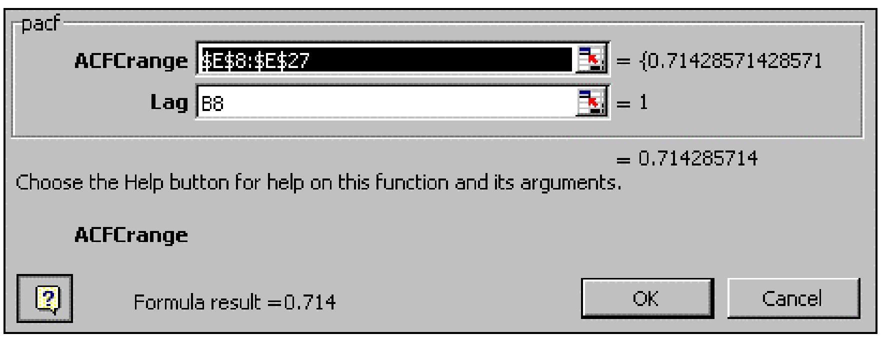
Hence

$$f_{33} = \frac{r_3 - \sum_{j=1}^{2} f_{2,j}\, r_{3-j}}{1 - \sum_{j=1}^{2} f_{2,j}\, r_j}$$

= [0.179 - (0.9375)(0.357)-(-0.313)(0.714)]/ [1 - (0.9375)(0.714)-(-0.313)(0.357)]

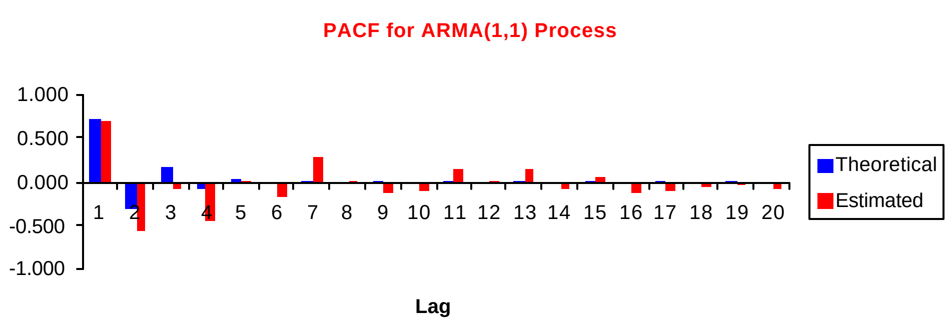
= 0.152

We can use the PACF function to calculate the theoretical PACF coefficients, using the *theoretical* ACF as the first input parameter. Click the Function Wizard and selection user defined functions, PACF. The input parameters are as shown below:

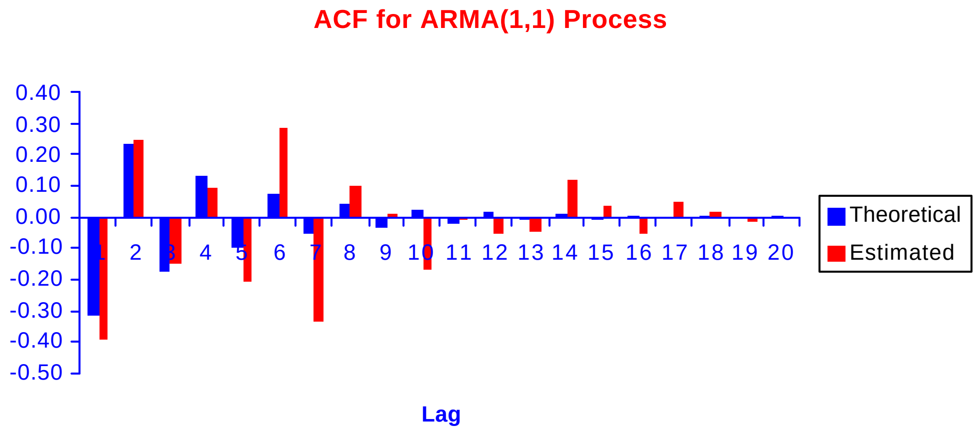


Likewise we can calculate the estimated PACF, this time using the *estimated* ACF as the first input parameter.

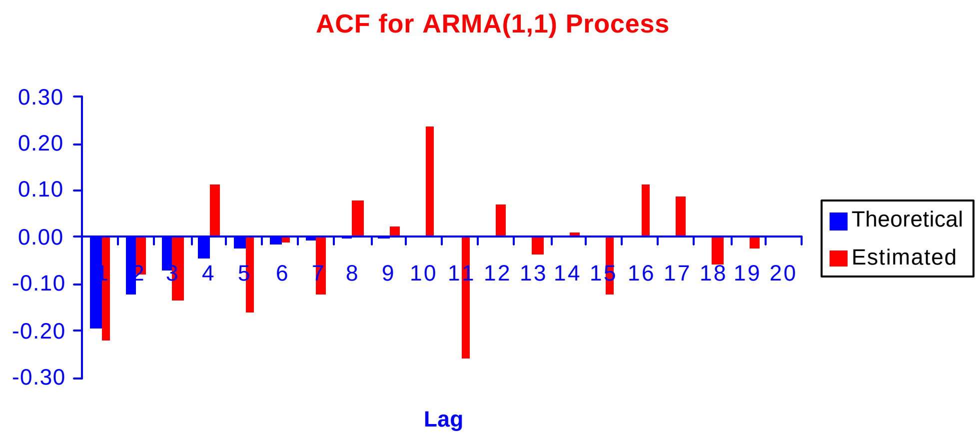
The resulting PACF correlogram is shown below.



5. With parameter $a_1 < 0$, the ACF follows an oscillating decay pattern, like the one below:



With parameter $a_1 > 0$, the ACF follows a geometric decay pattern, with positive or negative coefficients depending on whether $(a_1 + \beta_1) > 0$ or < 0. The rate of decay is governed by the coefficient a_1. The more positive the value, the slower the decay.
An example is shown below:



With parameter $\beta_1 < 0$, the PACF follows an exponential decay pattern, with sign = Sign $(a_1 + \beta_1)$, like the one below. With parameter $\beta_1 > 0$, the PACF follows an oscillating decay pattern (see example with $\beta_1 = 0.5$).